

Solution of the Matrix Hamiltonians via asymptotic iteration method

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Abstract

A method is suggested to obtain solutions of the various quantum optical Hamiltonians in the framework of the asymptotic iteration method. We extend the notion of asymptotic iteration method to solve the 2×2 matrix Hamiltonians. On a particular case, eigenvalues of the Rabi and Rashba Hamiltonians are computed. The method presented here reproduces a number of earlier results in a natural way as well as leads to a novel findings. Possible generalizations of the method are also suggested.

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1 Introduction

The solutions of the quantum optical Hamiltonians are an important theme in the existing literature [1, 2, 3, 4, 5]. There has been a great deal of interest in quantum optical models which reveal new physical phenomena described by the 2×2 matrix Hamiltonians. These Hamiltonians appear at the different fields of the physics [6, 7, 8, 9, 10, 11]. It is well known that the interaction of a two-level systems with a radiation field has a long varied history and these problems are often modelled by using 2×2 matrix Hamiltonians. As an example a physical system which describes the optical and electrical properties of confined electrons in semiconductor quantum wells, quantum dots and quantum wires depends on the Rashba spin orbit coupling and its equation is given by 2×2 matrix Hamiltonian. The Rashba splitting has been observed in many experiments and it constitutes the basis of the proposed electronic nano structures [12]. Due to the practical and technological importance of these models, it is not surprising that various aspects have been studied both analytically and numerically. Such systems have often been analyzed by using numerical methods because the implementation of the analytical techniques does not yield simple analytical expressions. Remarkably, exact solutions have not been thus far presented except for special cases even though it has been suggested that the problem may be solved exactly and their analytical treatments require tedious calculations [13, 14, 15, 16, 17, 18, 19, 20, 21].

In recent years much attention has been focused on asymptotic iteration method (AIM) [22, 23, 24, 25, 26, 27, 28]. This method reproduces exact solutions to many exactly solvable differential equations and these equations can be related to the Schrödinger equation. It also gives accurate results for the non solvable Schrödinger equation including sextic oscillator, cubic oscillator, deformed Coulomb potential etc., which are important in applications to many problems in physics. Encouraged by its satisfactory performance through comparisons with the other methods, we feel tempted to develop AIM to solve matrix differential equations. In contrast to the solution of the Schrödinger equation by using AIM including Coulomb, Morse, harmonic oscillator, etc. type potentials, the study of the quantum optical Hamiltonians has not attracted much attention in the literature. Such Hamiltonians have been found to be useful in the study of electronic properties of semiconductors, quantum dots and quantum wells.

The aim of this paper is to develop AIM for solving 2×2 matrix Hamiltonians and discuss their applications. Results of our procedure include the solutions of the Rabi [11] and Rashba [6] Hamiltonians. We provide a first step toward the extension of the technique to the solution of the various matrix Hamiltonians, whose spectrum can not be obtained exactly.

The paper is organized as follows. In section 2, we develop the AIM to obtain

eigenvalues and eigenfunctions of the wide range of the matrix Hamiltonians. Section 3 is devoted to solve the Rabi Hamiltonian in the framework of the AIM. In section 4 we present the solution of the Rashba Hamiltonian. In section 3 and 4 we also discuss the bosonisation of the physical Hamiltonians whose original forms are given as differential operators. We also present a procedure to transform the bosons in to Bargman-Fock space which is necessary to obtain first-order matrix differential Hamiltonians. The paper ends with a brief conclusion.

2 Formalism of the asymptotic iteration method for matrix Hamiltonians

The AIM is proposed to solve the second-order differential equations and the details can be found in [22]. In this section we systematically extend the method for the 2×2 first-order matrix differential equations. We begin by rewriting a first-order differential equation in the following matrix form:

$$I\phi' = u_0\phi \quad (1)$$

where $\phi = [\phi_1, \phi_2]^T$, two-component column vector u_0 is 2×2 matrix function and I is 2×2 unit matrix. Note that ϕ is the function of x and ϕ' is the first derivative with respect to x . Now, in order to obtain a general solution to this equation in the framework of the AIM we use the similar arguments given in [22]. More explicitly, the differential equation (1) can be written as the two coupled equations

$$\phi_1' = a_0\phi_1 + b_0\phi_2; \quad \phi_2' = c_0\phi_2 + d_0\phi_1 \quad (2)$$

where a_0, b_0, c_0 and d_0 are elements of the matrix u_0 . It is easy to show that n^{th} derivative of the ϕ_1 and ϕ_2 can be expressed as

$$\begin{aligned} \phi_1'' &= a_1\phi_1 + b_1\phi_2; & \phi_2'' &= c_1\phi_2 + d_1\phi_1 \\ \phi_1''' &= a_2\phi_1 + b_2\phi_2; & \phi_2''' &= c_2\phi_2 + d_2\phi_1 \\ &\dots & \\ \phi_1^{(n)} &= a_{n-1}\phi_1 + b_{n-1}\phi_2; & \phi_2^{(n)} &= c_{n-1}\phi_2 + d_{n-1}\phi_1 \\ \phi_1^{(n+1)} &= a_n\phi_1 + b_n\phi_2; & \phi_2^{(n+1)} &= c_n\phi_2 + d_n\phi_1. \end{aligned} \quad (3)$$

In order to discuss the asymptotic properties of (1), it is necessary to determine the coefficients a_n, b_n, c_n and d_n . After some straightforward calculation one can obtain the following relations:

$$\begin{aligned} a_n &= a_0a_{n-1} + a_{n-1}' + d_0b_{n-1} \\ b_n &= b_0a_{n-1} + b_{n-1}' + c_0b_{n-1} \\ c_n &= c_0c_{n-1} + c_{n-1}' + b_0d_{n-1} \\ d_n &= d_0c_{n-1} + d_{n-1}' + a_0d_{n-1}. \end{aligned} \quad (4)$$

Our task is now to introduce the asymptotic aspect of the method. For this purpose n^{th} and $(n+1)^{th}$ derivative of the ϕ_1 and ϕ_2 can be written as

$$\begin{aligned}\phi_1^{(n)} &= a_{n-1} \left(\phi_1 + \frac{b_{n-1}}{a_{n-1}} \phi_2 \right); & \phi_2^{(n)} &= c_{n-1} \left(\phi_2 + \frac{d_{n-1}}{c_{n-1}} \phi_1 \right) \\ \phi_1^{(n+1)} &= a_n \left(\phi_1 + \frac{b_n}{a_n} \phi_2 \right); & \phi_2^{(n+1)} &= c_n \left(\phi_2 + \frac{d_n}{c_n} \phi_1 \right).\end{aligned}\quad (5)$$

The coefficients a_0, b_0, d_0 and c_0 include the coupling constants. Therefore, for sufficiently large n we can suggest the following asymptotic constraints:

$$\frac{b_{n-1}}{a_{n-1}} = \frac{b_n}{a_n} = \gamma_1; \quad \frac{d_{n-1}}{c_{n-1}} = \frac{d_n}{c_n} = \gamma_2. \quad (6)$$

In this formalism the relations given in (6) imply that the wave function ϕ_1 and ϕ_2 are truncated for sufficiently large n and the roots of the relations (6) belong to the spectrum of the matrix Hamiltonian. Therefore one can easily compute the eigenenergies of the Hamiltonian by solving (6) for the energy term when $x \rightarrow x_0$.

Under the asymptotic condition given in (6), one can find the wave functions ϕ_1 and ϕ_2 . When we take $\frac{\phi_1^{(n+1)}}{\phi_1^{(n)}}$ and $\frac{\phi_2^{(n+1)}}{\phi_2^{(n)}}$ by using (5) under the constraints given in (6), we obtain:

$$\phi_1^{(n)} = \exp \left(\int \frac{a_n}{a_{n-1}} dx \right) \quad \text{or} \quad \phi_2^{(n)} = \exp \left(\int \frac{c_n}{c_{n-1}} dx \right). \quad (7)$$

Substituting the expression of a_n (and c_n) given in (4) into (7) and then replacing the $\phi_1^{(n)}$ (and $\phi_2^{(n)}$) in the first expression of (5), respectively, one gets the following expressions;

$$\phi_1 + \gamma_1 \phi_2 = \exp \left(\int (a_0 + \gamma_1 d_0) dx \right) \quad \text{or} \quad \phi_2 + \gamma_2 \phi_1 = \exp \left(\int (c_0 + \gamma_2 b_0) dx \right). \quad (8)$$

An immediate practical consequence of these results is that the eigenvalues and eigenfunctions of the various quantum optical Hamiltonians can easily be determined. We mention here that there exist numerous physical Hamiltonians which can be written in the form of the first-order matrix differential equation or can be transformed in the form of the first-order matrix differential equation. For example, the Dirac equation [25] is a first-order matrix differential equation while most of the Hamiltonians of the quantum optical systems can be written as first-order matrix differential equation in the Bargman-Fock space. Now we can determine the eigenvalues and eigenfunctions of the 2×2 matrix Hamiltonians. In the following sections, it will be shown that this asymptotic approach opens the way to treatment of a large class of matrix Hamiltonians of practical interest.

3 Solution of the Rabi Hamiltonian by using AIM

In this section we take a new look at the solution of Rabi Hamiltonian through the AIM. The Rabi Hamiltonian is a successful model of the interaction between matter and electromagnetic radiation. The Hamiltonian describes interacting a dipole interaction with a single mode of radiation in a two-level system. This Hamiltonian can be solved exactly within the framework of rotating wave approximation and its quasi-exact or isolated exact solutions can be obtained for some specific cases. Let us consider the Rabi Hamiltonian for a two-level atom [11]:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} I + \frac{1}{2} m \omega^2 x^2 I + \omega_0 \sigma_0 + \kappa (\sigma_+ + \sigma_-) x \quad (9)$$

where ω_0 is the atomic level splitting, ω is the frequency of the oscillation and κ is the coupling strength of the atom to the field and, $\sigma_0, \sigma_+, \sigma_-$ are

$$\sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (10)$$

Pauli-related matrices. Eigenvalues of the Hamiltonian (6) can be obtained from the eigenvalue equation $H\phi = E\phi$ where ϕ is a two component wave function. In general, it is difficult to determine the asymptotic aspects of the (9) in the present form. Therefore it is worth to transform (9) to an appropriate form. One way to transform a Hamiltonian in the form of the first-order matrix differential equation is to construct its bosonic representation. We are interested in the two-level system in one and two-dimensional geometry, whose Hamiltonians are given in terms of bosons-fermions or matrix-differential equations. Therefore, it is worth to express a suitable differential realizations of the bosons. Consider the following boson realizations

$$a = \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} + \sqrt{\frac{m\omega}{2\hbar}} x; \quad a^+ = -\sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} + \sqrt{\frac{m\omega}{2\hbar}} x. \quad (11)$$

They satisfy the usual commutation relation

$$[a, a^+] = 1; \quad [a, a] = [a^+, a^+] = 0 \quad (12)$$

In terms of the boson operators, the Hamiltonian (9) can be written as

$$H = \hbar\omega \left(a^+ a + \frac{1}{2} \right) I + \omega_0 \sigma_0 + \sqrt{\frac{\hbar}{2m\omega}} \kappa (\sigma_+ + \sigma_-) (a^+ + a). \quad (13)$$

It is interesting that this type Hamiltonian gives good results when it is solved by using the AIM in the Bargman-Fock space. In order to transform the Hamiltonian

to the Bargman-Fock space, we introduce the following transformation operator:

$$\Gamma_1 = \exp \left[\frac{\alpha_1}{2} (a^2 + a^{+2}) \right]. \quad (14)$$

The action of the operator on the bosons is given by

$$\Gamma_1 a \Gamma_1^{-1} = a \cos \alpha_1 - a^+ \sin \alpha_1; \Gamma_1 a^+ \Gamma_1^{-1} = a^+ \cos \alpha_1 + a \sin \alpha_1. \quad (15)$$

For the values $\alpha_1 = \pi/4$ boson operators take the form

$$a \rightarrow \frac{1}{\sqrt{2}} (a - a^+) = \sqrt{\frac{\hbar}{m\omega}} \frac{d}{dx}; \quad a^+ \rightarrow \frac{1}{\sqrt{2}} (a + a^+) = \sqrt{\frac{m\omega}{\hbar}} x \quad (16)$$

The final form of the Hamiltonian (13) with the realization (16) can be expressed as

$$H = \hbar\omega \left(x \frac{d}{dx} + \frac{1}{2} \right) I + \omega_0 \sigma_0 + \frac{\kappa}{\sqrt{2}} (\sigma_+ + \sigma_-) \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right). \quad (17)$$

It is obvious that the Hamiltonian (17) can easily be written as the two coupled first-order differential equation

$$\begin{aligned} \phi'_1 &= a_0 \phi_1 + b_0 \phi_2 \\ \phi'_2 &= c_0 \phi_2 + d_0 \phi_1. \end{aligned} \quad (18)$$

For the sake of simplicity let us take $\hbar = \omega = m = 1$ at that point, then the coefficients of the coupled differential equations are given by

$$\begin{aligned} a_0 &= \frac{x(2E - 1 + \kappa^2 - 2\omega_0)}{2x^2 - \kappa^2} \\ b_0 &= \frac{\kappa(1 - 2E - 2x^2 - 2\omega_0)}{\sqrt{2}(2x^2 - \kappa^2)} \\ c_0 &= \frac{x(2E - 1 + \kappa^2 + 2\omega_0)}{2x^2 - \kappa^2} \\ d_0 &= \frac{\kappa(1 - 2E - 2x^2 + 2\omega_0)}{\sqrt{2}(2x^2 - \kappa^2)}. \end{aligned} \quad (19)$$

Using a simple MATHEMATICA program one can compute a_n, b_n, c_n and d_n by the relations given in (4). On the other hand, for each iteration, the quantization condition $\delta_1(x) = b_{n-1}(x)a_n(x) - a_{n-1}(x)b_n(x)$ (and $\delta_2(x) = d_{n-1}(x)c_n(x) - c_{n-1}(x)d_n(x)$) depends on some variables namely E_n, κ, ω_0 and x . It is noticed that the iterations should be terminated by imposing the condition $\delta_i(x) = 0, i = 1, 2$ as an approximation to Eq. (6) to obtain the eigenenergies. The calculated eigenenergies E_n by means of this condition should, however, be independent of the choice of x . The choice of x is observed to be critical only to the speed of

the convergence of the eigenenergies, as well as for the stability of the process. In our study it has been observed that the optimal choice for x is the extremum point of the potential that is when $x = 0$. Therefore, we set $x = 0$ at the end of the iterations. Then, the roots of the iteration produce eigenenergies for the Rabi Hamiltonian. The results are reported in Table 1.

We can also obtain eigenfunctions of the Hamiltonian (9) by using the relation (8). Analytical expression for the ground state wave function of the Rabi Hamiltonian for some different parameters are given by

$$\begin{aligned} \kappa &= 0; \quad \omega_0 = 0; \quad E = n + \frac{1}{2}; \quad \phi = \begin{pmatrix} C_1 x^n \\ C_2 x^n \end{pmatrix}; \\ \kappa &= 0; \quad \omega_0 = \frac{1}{2}; \quad E = n; \quad \phi = \begin{pmatrix} C_1 x^{n-1} \\ C_2 x^n \end{pmatrix}; \\ \kappa &= \frac{1}{2}; \quad \omega_0 = 0; \quad E = n + \frac{3}{8}; \quad \phi = \begin{pmatrix} C_1 e^{-\frac{x}{2\sqrt{2}}} (1 + 2\sqrt{2}x)^n \\ C_1 e^{-\frac{x}{2\sqrt{2}}} (1 + 2\sqrt{2}x)^n \end{pmatrix}. \end{aligned} \quad (20)$$

The solution of this system describes a quantum mechanical state of H provided that ϕ belongs to the Bargman-Fock space. It is noticed that when κ or ω_0 is zero then the Hamiltonian (9) can be solved exactly. We have observed that for $n = 5$, accurate eigenvalues can be obtained after 15 iteration. As a consequence we have demonstrated that the solution of the Rabi Hamiltonian can be treated within the AIM. Our approach is relatively simple and gives accurate result.

4 Solution of the Rashba Hamiltonian by using AIM

The origin of the Rashba spin-orbit coupling in quantum dots is due to the lack of inversion symmetry which causes a local electric field perpendicular to the plane of heterostructure. The Hamiltonian representing the Rashba spin orbit coupling for an electron in a quantum dot can be expressed as [6]

$$H_R = \frac{\lambda_R}{\hbar} (p_y \sigma_x - p_x \sigma_y) \quad (21)$$

where λ_R represents the strength of the spin orbit coupling, and it can be adjusted by changing the asymmetry of the quantum well via external electric field and the matrices σ_x , and σ_y are Pauli matrices. Now we assume that the electron is confined in a parabolic potential

$$V = \frac{1}{2} m^* \omega_0^2 (x^2 + y^2) \quad (22)$$

here m^* is the effective mass of the electron and ω_0 is the confining potential frequency. The Hamiltonian describing an electron in two-dimensional quantum

dot takes the form

$$H = \frac{1}{2m^*} (P_x^2 + P_y^2) + \frac{1}{2}g\mu B\sigma_0 + V + H_R. \quad (23)$$

The term $\frac{1}{2}g\mu B\sigma_0$ introduces the Zeeman splitting between the (+) x -polarized spin up and (-) x -polarized spin down. The factors g is the gyromagnetic ratio and μ is the Bohr magneton. The kinetic momentum $\mathbf{P} = \mathbf{p} + e\mathbf{A}$ is expressed with canonical momentum $\mathbf{p} = -i\hbar(\partial_x, \partial_y, 0)$ and the vector potential \mathbf{A} can be related with the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The choice of symmetric gauge vector potential $\mathbf{A} = B/2(-y, x, 0)$, leads to the following Hamiltonian

$$\begin{aligned} H = & -\frac{\hbar^2}{2m^*} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}m^*\omega^2(x^2 + y^2) \\ & + \frac{1}{2}i\hbar\omega_c \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) + \frac{1}{2}g\mu B\sigma_0 + H_R \end{aligned} \quad (24)$$

where $\omega_c = eB/m^*$ stands for the cyclotron frequency of the electron, $\omega = \sqrt{\omega_0^2 + \left(\frac{\omega_c}{2}\right)^2}$ is the effective frequency. From now on we restrict ourselves to the solution of (24).

4.1 Bosonisation of the Hamiltonian

The Hamiltonian (24) can not be solved within the framework of the AIM in the present form. Our task is now to demonstrate that the Hamiltonian (24) can be expressed as two coupled first-order differential equation in the Bargman-Fock space. One way to express the Hamiltonian H with boson operators is to introduce an appropriate differential realization for bosons. It can easily be bosonised when the boson operators are realized as

$$\begin{aligned} a^+ &= \sqrt{\frac{m^*\omega}{4\hbar}}(x + iy) - \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x + i\partial_y) \\ a &= \sqrt{\frac{m^*\omega}{4\hbar}}(x - iy) + \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x - i\partial_y) \\ b^+ &= \sqrt{\frac{m^*\omega}{4\hbar}}(x - iy) - \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x - i\partial_y) \\ b &= \sqrt{\frac{m^*\omega}{4\hbar}}(x + iy) + \sqrt{\frac{\hbar}{4m^*\omega}}(\partial_x + i\partial_y). \end{aligned} \quad (25)$$

They satisfy the usual commutation relations. Insertion of (25) into (24) yields the following Hamiltonian:

$$H = \hbar\omega(a^+a + b^+b + 1) + \frac{\hbar\omega_c}{2}(a^+a - b^+b)$$

$$-\sqrt{\frac{m^*\omega}{\hbar}}\lambda_R \left[(b^+ - a)\sigma_+ + (b - a^+)\sigma_- \right] + \frac{1}{2}g\mu B\sigma_0 \quad (26)$$

Now we turn our attention to the transformation of the Hamiltonian in the form of the first-order one variable matrix differential equation. A simple connection between the Hilbert space and the Bargman-Fock space can be obtained by transforming the differential realizations of the creation and annihilation operators (25). This can be done by introducing the following similarity transformation operators

$$\Gamma_2 = \exp \left[\frac{\alpha_2}{2} (b^2 + b^{+2}) \right]; \quad \Lambda = \exp \left[\beta (a^+b + b^+a) \right]. \quad (27)$$

The operators act on the bosons as follows:

$$\begin{aligned} \Gamma_1 a \Gamma_1^{-1} &= a \cos \alpha_1 - a^+ \sin \alpha_1; & \Gamma_1 a^+ \Gamma_1^{-1} &= a^+ \cos \alpha_1 + a \sin \alpha_1 \\ \Gamma_2 b \Gamma_2^{-1} &= a \cos \alpha_2 - a^+ \sin \alpha_2; & \Gamma_2 a^+ \Gamma_2^{-1} &= a^+ \cos \alpha_2 + a \sin \alpha_2 \\ \Lambda a \Lambda^{-1} &= a \cos \beta - b \sin \beta; & \Lambda a^+ \Lambda^{-1} &= a^+ \cos \beta - b^+ \sin \beta \\ \Lambda b \Lambda^{-1} &= b \cos \beta + a \sin \beta; & \Lambda b^+ \Lambda^{-1} &= b^+ \cos \beta + a^+ \sin \beta \end{aligned} \quad (28)$$

These transformations play a key role to construct one variable first-order matrix differential equation form of (26). The similar transformation and the change of the variable $y \rightarrow iy$ gives the following realizations

$$\begin{aligned} a &\rightarrow \Lambda \Gamma_1 a \Gamma_1^{-1} \Lambda^{-1} = \sqrt{\frac{\hbar}{m^*\omega}} \frac{\partial}{\partial x}, & a^+ &\rightarrow \Lambda \Gamma_1 a^+ \Gamma_1^{-1} \Lambda^{-1} = \sqrt{\frac{m^*\omega}{\hbar}} x \\ b &\rightarrow \Lambda \Gamma_2 b \Gamma_2^{-1} \Lambda^{-1} = \sqrt{\frac{\hbar}{m^*\omega}} \frac{\partial}{\partial y}, & b^+ &\rightarrow \Lambda \Gamma_2 b^+ \Gamma_2^{-1} \Lambda^{-1} = \sqrt{\frac{m^*\omega}{\hbar}} y. \end{aligned} \quad (29)$$

It is obvious that the Hamiltonian (26) can be put in the form of the first-order matrix differential with the realization of bosons (29), but it consists of two variables x and y . In order to separate the variables consider the following conserved quantity of the Hamiltonian (26):

$$K = a^+a - b^+b - \frac{1}{2}\sigma_0. \quad (30)$$

If K and H commute, the eigenfunction of K is also eigenfunction of the H . Therefore it is worth to obtain eigenfunction of K . When we solve the following eigenvalue equation

$$K |n_1, n_2\rangle = \left(k + \frac{1}{2}\right) |n_1, n_2\rangle, \quad (31)$$

in the Bargman-Fock space we obtain the following expression:

$$\psi(x, y) = x^k \phi(xy) |\uparrow\rangle + x^{k+1} \phi(xy) |\downarrow\rangle. \quad (32)$$

where $|\uparrow\rangle$ stands for up state and $|\downarrow\rangle$ stands for down state. The eigenfunction of the Hamiltonian can be obtained from the relation

$$|n_1, n_2\rangle = \Gamma_2^{-1} \Gamma_1^{-1} \Lambda^{-1} \psi(x, y). \quad (33)$$

Substitution of (32) into the Hamiltonian (26) by using the realization of bosons in (29) leads to the following set of one variable coupled differential equations

$$\begin{aligned} \hbar\omega \left[2z \frac{d}{dz} + k + 1 + \frac{k\omega_c}{2\omega} - \frac{\mu g B}{2\hbar\omega} - \frac{E}{\hbar\omega} \right] \phi_1(z) \\ + \lambda_R \left[k + 1 - \frac{m^*\omega z}{\hbar} + z \frac{d}{dz} \right] \phi_2(z) = 0 \end{aligned} \quad (34)$$

and

$$\begin{aligned} \hbar\omega \left[2z \frac{d}{dz} + k + 2 + \frac{(k+1)\omega_c}{2\omega} + \frac{\mu g B}{2\hbar\omega} - \frac{E}{\hbar\omega} \right] \phi_2(z) \\ + \lambda_R \left[\frac{m^*\omega}{\hbar} - \frac{d}{dz} \right] \phi_1(z) = 0 \end{aligned} \quad (35)$$

where $z = xy$ and E is the eigenvalues of the Hamiltonian H and $\phi_1(z)$ and $\phi_2(z)$ correspond up and down eigenstates of the Hamiltonian H , respectively. Following the analysis of [8] which was constructed to obtain the solution of the Rashba Hamiltonian, one can obtain the quasi-exact solution of the differential equations. Here we present solution of the problem in the framework of the AIM.

4.2 Solution of the Rashba Hamiltonian

In the previous section we have formulated the Rashba Hamiltonian based on the two boson operators and we have discussed its transformation to the one variable differential equation. In this section we present the AIM which leads to solution of the Hamiltonian (26). For the sake of simplicity let us take $\hbar = m^* = \omega_0 = 1$, then the last expression given above can easily be written as the two coupled differential equation

$$\begin{aligned} \phi_1' &= a_0 \phi_1 + b_0 \phi_2 \\ \phi_2' &= c_0 \phi_2 + d_0 \phi_1 \end{aligned} \quad (36)$$

where

$$\begin{aligned} a_0 &= \frac{\omega(2E + \lambda_R^2 + Bg\mu - \omega_c k - 2\omega(k+1))}{(4\omega^2 z + \lambda_R^2)} \\ b_0 &= \frac{\lambda_R(-2E + Bg\mu + \omega_c(k+1) - 2\omega k + 4\omega^2 z)}{2(4\omega^2 z + \lambda_R^2)} \end{aligned}$$

$$\begin{aligned}
c_0 &= \frac{\omega z (2E - Bg\mu - \omega_c(k+1) - 2\omega(k+2)) - \lambda_R^2(1+k-\omega z)}{z(4\omega^2 z + \lambda_R^2)} \\
d_0 &= \frac{\lambda_R(2E + Bg\mu - \omega_c k - 2\omega(k+1) - 4\omega^2 z)}{2z(4\omega^2 z + \lambda_R^2)}
\end{aligned} \tag{37}$$

As in the previous section, one can compute a_n, b_n, c_n and d_n using the relations given in (4) by a simple MATHEMATICA program. When $z = 0$, the eigenvalues are produced by the roots of the quantization condition. The computed results of the iteration are reported in Table 2.

As it is seen that the energy eigenvalue equation is easily obtained by using AIM. This is the advantage of the AIM that gives the eigenvalues directly by transforming the quantum optical Hamiltonians in the form of the 2×2 matrix differential equation form. In a similar manner as in the previous section one can also obtain eigenfunction of the Hamiltonian (34) and (35) by using the relation (8).

5 Conclusion

We study the AIM to solve the problem of an electron in a quantum dot in the presence of both magnetic field and spin-orbit coupling. Our formulation gives an accurate result for the eigenvalues of Rabi and Rashba Hamiltonians. The suggested approach can easily be modified to solve some other quantum optical problems.

Furthermore we have presented a transformation procedure that offers several advantageous, especially if one wishes to describe the eigenvalues of the bosonic Hamiltonians by using AIM. It is obvious that the technique presented here have been used in a variety of problems to compute their spectrums. We have presented the steps towards an extension of the AIM.

The technique given in this article can be extended in several ways. The Hamiltonian of a quantum dot including position dependent effective mass may be formulated and solved within the procedure given here. We hope that our method leads to interesting results on the spin-orbit effects in quantum dots in future research. Success of our analysis leads to the solution of the wide range of physical systems.

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Table 1: Eigenvalues of the Rabi Hamiltonian for different values of energy state n , splitting constant ω_0 , and coupling constant κ .

n	ω_0	$\kappa=0.0$	$\kappa=0.25$	$\kappa=0.50$	$\kappa=0.75$	$\kappa=1.0$
0	0.0	0.5	0.46875	0.375	0.21875	0.0
		0.5	0.46875	0.375	0.21875	0.0
	0.5	0.0	-0.015748	-0.064513	-0.151195	-0.284922
		1.0	0.8080812	0.5865567	0.3344402	0.0488330
	1.0	-0.5	-0.510489	-0.542870	-0.600162	-0.688478
		0.5	0.4494499	0.3117158	0.1101679	-0.141057
1	0.0	1.5	1.46875	1.375	1.21875	1.0
		1.5	1.46875	1.375	1.21875	1.0
	0.5	2.0	1.7359493	1.4482320	1.1416351	0.8206445
		1.0	1.1601606	1.2791925	1.3249343	0.0488339
	1.0	1.5	1.4124109	1.1964923	0.9129976	0.5913838
		1.5	1.5294031	1.5980360	1.6078394	1.2925094
2	0.0	2.5	2.46875	2.375	2.21875	2.0
		2.5	2.46875	2.375	2.21875	2.0
	0.5	2.0	2.2317488	2.4052487	2.4171516	2.1405752
		3.0	2.6813089	2.3498342	2.0376015	1.8229775
	1.0	2.5	2.5660788	2.6980532	2.5296180	2.0954608
		2.5	2.3778828	2.1020231	1.8333934	1.8464607
3	0.0	3.5	3.46875	3.375	3.21875	3.0
		3.5	3.46875	3.375	3.21875	3.0
	0.5	4.0	3.6359015	3.2763600	3.0202409	2.9667546
		3.0	3.2858045	3.4872316	3.4040837	3.0257950
	1.0	3.5	3.3455073	3.0291733	2.9681181	3.1101317
		3.5	3.6001795	3.7530103	3.4089357	2.9237568
4	0.0	4.5	4.46875	4.375	4.21875	4.0
		4.5	4.46875	4.375	4.21875	4.0
	0.5	4.0	4.3305823	4.5395425	4.347928	3.9264039
		5.0	4.5965321	4.2235326	4.070904	4.0877781
	1.0	4.5	4.6320470	4.7466147	4.294056	3.7849395
		4.5	4.3150285	3.9930696	4.121917	4.2669272
5	0.0	5.5	5.46875	5.375	5.21875	5.0
		5.5	5.46875	5.375	5.21875	5.0
	0.5	6.0	5.5615693	5.1903384	5.1469657	5.1455854
		5.0	5.3692738	5.5672925	5.2808323	4.8635200
	1.0	5.5	5.2862655	5.0134462	5.2558821	5.2799593
		5.5	5.6619147	5.7022888	5.1889935	4.7100710

Table 2: Eigenvalues of the Rashba Hamiltonian for different values of external magnetic field magnitude, B , energy state n , and coupling constant κ .

k	B	n	$\kappa = 0.0$	$\kappa = 0.25$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 1.0$
0	0	0	0.0	0.0	0.0	0.0	0.0
		1	1.0	0.9393054	0.7737872	0.5320678	0.2330309
		2	2.0	1.9962478	1.9465668	1.7810714	1.4877948
		3	3.0	2.9429223	2.8214396	2.7072634	2.5694541
		4	4.0	3.9928041	3.9101840	3.6989847	3.4345727
		5	5.0	4.9462371	4.8526593	4.7537468	4.5359330
	3/2	0	1.5	1.5	1.5	1.5	1.5
		1	0.5	0.4776216	0.4097986	0.2944479	0.1280664
		2	3.0	2.8877702	2.6235301	2.2882565	1.9132967
		3	4.0	4.0892712	4.2713119	4.2790928	3.8467145
		4	5.5	5.3107639	4.9316662	4.6839650	4.8228044
		5	6.5	6.6655604	6.9171192	6.5731725	6.0024178
1	0	0	-1.0	-1.0	-1.0	-1.0	-1.0
		1	2.0	1.8847720	1.6082091	1.2505849	0.8439143
		2	3.0	3.0463969	3.0303266	2.7527743	2.3150943
		3	4.0	3.8955162	3.7596438	3.7797919	3.6831314
		4	5.0	5.0363268	4.9398058	4.6186235	4.3781510
		5	6.0	5.9052818	5.8399949	5.8328977	5.5201455
	3/2	0	1.0	1.0	1.0	1.0	1.0
		1	2.5	2.4555260	2.3240428	2.1107652	1.8227033
		2	5.0	4.8690809	4.5668665	4.1837412	3.7484901
		3	6.0	6.1078281	6.3230999	6.2570599	5.7506451
		4	7.5	7.2943805	6.8910658	6.7283380	6.9481530
		5	8.5	8.6817766	8.9384866	8.5213644	7.9201055